Due Fri

2.2 – Evaluating Determinants by Row Reduction

Theorem 2.2.1 Let A be a square matrix. If A has a row of zeros or a column of zeros, then $\det(A) = 0$.

Theorem 2.2.2 Let A be a square matrix. Then $\det(A) = \det(A^T)$.

Theorem 2.2.3 Elementary Row Operations

Let A be an $n \times n$ matrix.

a) If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then $\det(B) = k\det(A)$.

b) If B is the matrix that results when two rows or two columns of A are interchanged, then $\det(B) = -\det(A)$.

c) If B is the matrix that results when a multiple of one row of A is added to another or when a multiple of one column is added to another, then $\det(B) = \det(A)$.

Pf (a): Let B be the matrix that results from multiplying the i^{th} row of A by k.

$$\det(A) = \sum_{l=1}^{n} a_{il} C_{il} \text{ by expanding along the } i^{th} \text{ row}$$

$$\det(B) = \sum_{l=1}^{n} b_{il} C_{il} = \sum_{l=1}^{n} k a_{il} C_{il} = k \sum_{l=1}^{n} a_{il} C_{il}$$

$$= k \det(A)$$

Consider $\begin{vmatrix} 2 & 1 & 3 \\ 4 & 6 & 8 \\ 5 & 7 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 & 3 \\ 2 & 3 & 4 \\ 5 & 7 & 2 \end{vmatrix}$

Theorem 2.2.4 Let E be an $n \times n$ elementary matrix.

a) If E results from multiplying a single row of I_n by a nonzero number k, then $\det(E) = k$.

b) If E results from interchanging two rows of I_n, then $\det(E) = -1$.

c) If E results from adding a multiple of one row of I_n to another, then $\det(E) = 1$.

12. Evaluate the determinant of the matrix by first reducing the matrix to row echelon form and then using some combination of row operations and cofactor expansion.

$$\begin{bmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix}$$

$R_2 \to R_2 + 2R_1$

$$\begin{array}{ccc} -2 & 4 & 1 \\ 2 & -6 & 0 \\ \hline 0 & -2 & 1 \end{array}$$

$R_3 \to R_3 - 5R_1$

$$\begin{array}{ccc} 5 & -2 & 2 \\ -5 & 15 & 0 \\ \hline 0 & 13 & 2 \end{array}$$

$$\begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix}$$

$R_3 \to R_3 + \frac{13}{2} R_2$

$$\begin{array}{ccc} 0 & 13 & 2 \\ 0 & -13 & \frac{13}{2} \\ \hline 0 & 0 & \frac{17}{2} \end{array}$$

$$\begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & \frac{17}{2} \end{vmatrix}$$

$$\begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & \frac{17}{2} \end{vmatrix} = (-2)\left(\frac{17}{2}\right) \begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{vmatrix} = -17$$

Summary:

$$\begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & \frac{17}{2} \end{vmatrix} = (-2)\left(\frac{17}{2}\right) \begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{vmatrix}$$

That is, $\begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} = (-2)\left(\frac{17}{2}\right) \begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{vmatrix}$

Q: Is this true: $\begin{bmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{bmatrix} = (-2)\left(\frac{17}{2}\right) \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$

NO☹!!

In #16 and 20, evaluate the determinant, given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6.$$

16. $\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} = 6$

20. $\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix} = -12$

$R_2 \to 2R_2$

$R_3 \to R_3 + 3R_1$ does not change the determinant

$$\begin{vmatrix} a & b & c \\ \cancel{d} & \cancel{e} & \cancel{f} \\ g & h & i \end{vmatrix} = -2d\,\big|\;\big| + 2e\,\big|\;\big| - 2f\,\big|\;\big| \qquad \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix}$$

Theorem 2.2.5 If A is a square matrix with two proportional rows or two proportional columns, then $\det(A) = 0$.

One row or column operation can yield a row or column of zeros.